

A generalization of a theorem of Hoffman

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Abstract

In 1977, Hoffman gave a characterization of graphs with smallest eigenvalue at least -2 . In this paper we generalize this result to graphs with smaller smallest eigenvalue. For the proof, we use the concept of Hoffman graphs, as introduced by Woo and Neumaier in 1995. Our result says that if a graph with smallest eigenvalue at least $-t - 1$ (where t is a positive integer) satisfies some local conditions, then it is highly structured. We apply our result to graphs which are cospectral with the Hamming graph $H(3, q)$, the Johnson graph $J(v, 3)$ and the 2-clique extensions of grids, respectively.

Keywords : smallest eigenvalue, Hoffman graph, Johnson graph, Hamming graph, 2-clique extension of grid graph, intersection graph, hypergraph

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1 Introduction

In 1976, Cameron et al. [2] showed that a connected graph with smallest eigenvalue at least -2 is a generalized line graph if the number of vertices is at least 37. For the proof, the classification of the irreducible root lattices was essential. In 1977, Hoffman [5] showed the following result. Note that in this paper we will denote by $d(x)$ the degree of x .

Theorem 1.1. *Let $-1 - \sqrt{2} < \lambda \leq -2$ be a real number. Then there exists an integer $f(\lambda) > 0$, such that, if G is a graph satisfying*

- (i) $d(x) \geq f(\lambda)$ holds for all $x \in V(G)$;
- (ii) $\lambda \leq \lambda_{\min}(G) \leq -2$,

then G is a generalized line graph and $\lambda_{\min}(G) = -2$.

Note that Hoffman did not use the classification of the irreducible root lattices for the proof of this theorem. But this meant that he had to assume large minimum degree. In this paper, we will use ideas from Hoffman [5] to show four structural results that generalize Theorem 1.1. (For definitions, see next section.) In the rest of this paper, G_x denotes the induced subgraph on the neighbors of x in the graph G , and $\bar{d}(G_x)$ denotes the average degree of G_x . The first two results give sufficient conditions such that graph G is the intersection graph of some linear $(t+1)$ -uniform hypergraph.

Theorem 1.2. *Let $t \geq 2$ be an integer. Then, there exists a positive integer $K(t)$, such that if a graph G satisfies the following conditions:*

- (i) $d(x) > K(t)$ holds for all $x \in V(G)$;
- (ii) Any $(t^2 + 1)$ -plex containing x has order at most $\frac{d(x) - K(t)}{t}$ for all $x \in V(G)$;
- (iii) $\lambda_{\min}(G) \geq -t - 1$,

then G is the intersection graph of some linear $(t+1)$ -uniform hypergraph.

Theorem 1.3. *Let $t \geq 2$ be an integer. Then, there exists a positive integer $\kappa(t)$, such that if a graph G satisfies the following conditions:*

- (i) $d(x) > \kappa(t)$ holds for all $x \in V(G)$;

- (ii) $\bar{d}(G_x) \leq \frac{d(x) - \kappa(t)}{t}$ holds for all $x \in V(G)$;
- (iii) $\lambda_{\min}(G) \geq -t - 1$,

then G is the intersection graph of some linear $(t+1)$ -uniform hypergraph.

In the next two results, we will focus on graphs with smallest eigenvalue at least -3 and show the following results.

Theorem 1.4. *There exists a positive integer K , such that if a graph G satisfies the following conditions:*

- (i) $d(x) > K$ holds for all $x \in V(G)$;
- (ii) Any 5-plex containing x has order at most $d(x) - K$ for all $x \in V(G)$;
- (iii) $\lambda_{\min}(G) \geq -3$,

then G is the slim graph of a 2-fat $\{\text{A}, \text{B}, \text{C}\}$ -line Hoffman graph.

Theorem 1.5. *There exists a positive integer κ , such that if a graph G satisfies the following conditions:*

- (i) $d(x) > \kappa$ holds for all $x \in V(G)$;
- (ii) $\bar{d}(G_x) \leq d(x) - \kappa$ holds for all $x \in V(G)$;
- (iii) $\lambda_{\min}(G) \geq -3$,

then G is the slim graph of a 2-fat $\{\text{A}, \text{B}, \text{C}\}$ -line Hoffman graph.

We will give the following three results as applications of Theorems 1.2–1.5.

Theorem 1.6. *There exists a positive integer q' such that for each $q \geq q'$, any graph, that is cospectral with the Hamming graph $H(3, q)$ is the intersection graph of some linear 3-uniform hypergraph.*

Theorem 1.7. *There exists a positive integer v' such that for each $v \geq v'$, any graph, that is cospectral with the Johnson graph $J(v, 3)$ is the intersection graph of some linear 3-uniform hypergraph.*

Theorem 1.8. *There exists a positive integer t such that for each pair of (t_1, t_2) with $t_1 \geq t_2 \geq t$, any graph, that is cospectral with the 2-clique extension of $(t_1 \times t_2)$ -grid is the slim graph of a 2-fat $\{\text{A}, \text{B}, \text{C}\}$ -line Hoffman graph .*

Remark 1.9. (1) *Theorem 1.6 was first shown by S. Bang et al. [1]. They also obtained that $q' \leq 36$.*

(2) *E.R. van Dam et al. [11] gave two constructions for non-distance-regular graphs with the same spectrum as $J(v, 3)$ for all v . One method used Godsil and McKay switching and the other method was to construct such graphs as the point graph of a partial linear space. Theorem 1.7 states that if v is large enough, these graph have to be the point graphs of partial linear spaces.*

(3) *In a follow-up paper [13], we will use Theorem 1.8 to show that the 2-clique extension of the $(t \times t)$ -grid is determined by its spectrum if t is large enough.*

To show Theorems 1.2–1.5, we will use the theory of Hoffman graphs as introduced by Woo and Neumaier [12]. In Section 2, we will give definitions and basic theory of Hoffman graphs. In Section 3, we will define a set of matrices $\mathcal{M}(t)$ and a finite family of t -fat Hoffman graph $\mathfrak{G}(t)$, where t is a positive integer. In Corollary 3.8, we will use the set of matrices, $\mathcal{M}(t)$, to show that a t -fat Hoffman graph with smallest eigenvalue at least $-t - 1$ is a t -fat $\mathfrak{G}(t)$ -line Hoffman graph. In Section 4, we will consider associated Hoffman graphs of graphs, (as introduced by Kim et al. [7]) and show some properties for these Hoffman graphs we will use later. In Theorem 5.2 and Theorem 5.3, we will show that if a graph G satisfies some local conditions and has smallest eigenvalue at least $-t - 1$, then its associated Hoffman graph is a $\mathfrak{G}(t)$ -line Hoffman graph. This implies that G is highly structured. Also, in Section 5, we will show Theorem 1.2 and Theorem 1.4 (resp. Theorem 1.3 and Theorem 1.5) as a consequence of the Theorem 5.2 (resp. Theorem 5.3). In the last section, we will give proofs of Theorems 1.6, 1.7 and 1.8, by using Theorems 1.2–1.5.

2 Definitions and preliminaries

In this section, we will give definitions and preliminaries of Hoffman graphs.

2.1 Graphs and h -uniform hypergraphs

All the graphs considered in this paper are finite, undirected and simple. For a given graph $G = (V(G), E(G))$, the *adjacency matrix* $A(G)$ of G is the $(0, 1)$ -matrix with rows and columns indexed by the vertices of G such that the uv -entry of $A(G)$ is equal to 1 if and only if u and v are adjacent in G . The eigenvalues of G are the eigenvalues of $A(G)$. The *spectrum* of G is the multiset

$$\{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_t^{m_t}\},$$

where $\lambda_0, \lambda_1, \dots, \lambda_t$ are the distinct eigenvalues of G and m_i is the multiplicity of λ_i ($i = 0, 1, \dots, t$). Two graphs are called *cospectral* if they have the same spectrum.

A graph G is called *walk-regular* if the number of walks of length r from a given vertex x to itself (closed walk) is independent of the choice of x , for all r . Since this number equals A_{xx}^r , it is the same as saying that A^r has constant diagonal for all r , where $A = A(G)$ is the adjacency matrix of G . Note that a walk-regular graph is always regular.

For the following discussion, we follow E.R. van Dam [10].

Let G be a connected k -regular graph with n vertices and adjacency matrix A . Suppose that G has exactly four distinct eigenvalues $\lambda_0 = k, \lambda_1, \lambda_2, \lambda_3$, then $(A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I) = \frac{1}{n}(k - \lambda_1)(k - \lambda_2)(k - \lambda_3)J$, where J is the all-one matrix. Since A^2, A, I and J all have constant diagonal, we see that A^r has constant diagonal for every r . So G is walk-regular. In particular, A_{xx}^3 , which counts twice the number of triangles through x , does not depend on the vertex x and equals $(\lambda_1 + \lambda_2 + \lambda_3)k + \lambda_1 \lambda_2 \lambda_3 + \frac{1}{n}(k - \lambda_1)(k - \lambda_2)(k - \lambda_3)$. Hence, for any vertex x in G , the local graph of G at x has average degree

$$\bar{d}(G_x) = \lambda_1 + \lambda_2 + \lambda_3 + \frac{1}{k} \lambda_1 \lambda_2 \lambda_3 + \frac{1}{nk} (k - \lambda_1)(k - \lambda_2)(k - \lambda_3). \quad (1)$$

In [9], a generalization of a clique is introduced as follows:

Definition 2.1. *Let p be a positive integer. A p -plex is an induced subgraph in which each vertex is adjacent to all but at most p of the vertices.*

Note that a clique is exactly the same as a 1-plex.

A *hypergraph* H is a pair (X, E) , where the elements of E are non-empty subsets (of any cardinality) of the finite set X . The set X is called the vertex set of H and E is called the (hyper)edge set of H . The hypergraph

$H = (X, E)$ is said *h-uniform* if each $e \in E$ has size h where h is an integer at least 2, and is *linear* if every pair of distinct vertices of H is contained in at most one edge of H .

Definition 2.2. Suppose $H = (X, E)$ is a hypergraph. The intersection graph of H , denoted by $IG(H)$, is the graph where $V(IG(H)) = E$ and $E(IG(H))$ is the set of all unordered pairs $\{e, e'\}$ of distinct elements of E such that $e \cap e' \neq \emptyset$ in H .

Clearly, a 2-uniform hypergraph is a graph.

2.2 Matrices and interlacing

Suppose that M_1 is a real symmetric $n \times n$ matrix, and that M_2 is a real symmetric $m \times m$ matrix ($m \leq n$). Let $\theta_1(M_1) \geq \theta_2(M_1) \geq \dots \geq \theta_n(M_1)$ and $\theta_1(M_2) \geq \theta_2(M_2) \geq \dots \geq \theta_m(M_2)$ denote their eigenvalues in nonincreasing order. We say that the eigenvalues of M_2 *interlace* the eigenvalues of M_1 if for $i = 1, \dots, m$,

$$\theta_{n-m+i}(M_1) \leq \theta_i(M_2) \leq \theta_i(M_1).$$

By [3, Theorem 9.1.1] and [3, Lemma 9.6.1], we have the following interlacing result:

Lemma 2.3. (i) Suppose M is a real symmetric matrix and M' is any principal submatrix of M , the eigenvalues of M' interlace the eigenvalues of M .

(ii) Suppose G is a graph and let π be a partition of $V(G)$ with cells V_1, \dots, V_r . Define the quotient matrix of $A(G)$ relative to π to be the $r \times r$ matrix B such that the ij -entry of B is the average number of neighbors in V_j of vertices in V_i . Then the eigenvalues of B interlace the eigenvalues of $A(G)$.

The following result is an upper bound of the order of a $(p+1)$ -plex for a regular graph. It is a generalization of the Hoffman bound on independent sets for regular graphs (see, [3, Lemma 9.6.2]).

Theorem 2.4. Let G be a k -regular graph with n vertices and let $\theta_1 = k \geq \theta_2 \geq \dots \geq \theta_n$ be the eigenvalues of G . Then the order of a $(p+1)$ -plex is at most $\frac{n(p+1+\theta_2)}{n-k+\theta_2}$.

Proof. Let P be a $(p+1)$ -plex of G with $|V(P)| = m$. Consider the partition $\pi = \{V(P), V(G) - V(P)\}$ of $V(G)$. The quotient matrix B of $A(G)$ relative to π is

$$B = \begin{pmatrix} \alpha & k - \alpha \\ \beta & k - \beta \end{pmatrix},$$

where α and β are real numbers satisfying $\alpha \geq m - (p+1)$ and $\beta = \frac{m(k-\alpha)}{n-m}$. Then B has eigenvalues k and $\alpha - \beta$, as $\text{tr } B = k + \alpha - \beta$. By Lemma 2.3 (ii), we have $m - (p+1) - \frac{m(k-(m-(p+1)))}{n-m} \leq \alpha - \beta \leq \theta_2$. This completes the proof. \square

2.3 Hoffman graphs and special matrices

In this subsection we introduce Hoffman graphs and their special matrices.

Definition 2.5. A Hoffman graph \mathfrak{h} is a pair (H, ℓ) , where $H = (V, E)$ is a graph and $\ell : V \rightarrow \{f, s\}$ is a labeling map on V , such that any two vertices with label f are not adjacent and every vertex with label f has at least one neighbor with label s .

The vertices with label f are called *fat vertices*, and the vertices with label s are called *slim vertices*. We denote the set of slim vertices by $V_s(\mathfrak{h})$ and the set of fat vertices by $V_f(\mathfrak{h})$. For a vertex x of \mathfrak{h} , the set of slim (resp. fat) neighbors of x in \mathfrak{h} is denoted by $N_{\mathfrak{h}}^s(x)$ (resp. $N_{\mathfrak{h}}^f(x)$). A Hoffman graph is called *t-fat* if every slim vertex has at least t fat neighbors, where t is a positive integer. A *fat* Hoffman graph is a 1-fat Hoffman graph.

The *slim graph* of \mathfrak{h} is the subgraph of H induced by $V_s(\mathfrak{h})$. Note that we consider the slim graph of a Hoffman graph as an ordinary graph.

Definition 2.6. Suppose $\mathfrak{h} = (H, \ell)$ is a Hoffman graph.

- (i) The Hoffman graph $\mathfrak{h}_1 = (H_1, \ell_1)$ is called an induced Hoffman subgraph of \mathfrak{h} , if H_1 is an induced subgraph of H and $\ell_1(x) = \ell(x)$ for all vertices x of H_1 .
- (ii) Let W be a subset of $V_s(\mathfrak{h})$. The induced Hoffman subgraph of \mathfrak{h} generated by W , denoted by $\langle W \rangle_{\mathfrak{h}}$, is the Hoffman subgraph of \mathfrak{h} induced by $W \cup \{f \in V_f(\mathfrak{h}) \mid f \sim w \text{ for some } w \in W\}$.

Definition 2.7. Two Hoffman graphs $\mathfrak{h} = (H, \ell)$ and $\mathfrak{h}' = (H', \ell')$ are called isomorphic to \mathfrak{h} if there exists an isomorphism from H to H' which preserves the labeling.

For a Hoffman graph $\mathfrak{h} = (H, \ell)$, let A be the adjacency matrix of H

$$A = \begin{pmatrix} A_s & C \\ C^T & O \end{pmatrix}$$

in a labeling in which the fat vertices come last. The real symmetric matrix $S(\mathfrak{h}) := A_s - CC^T$ is called the *special matrix* of \mathfrak{h} . *Eigenvalues* of \mathfrak{h} are the eigenvalues of $S(\mathfrak{h})$.

Note that Hoffman graphs are not determined by their special matrices.

Example 2.8. *(Two non-isomorphic Hoffman graphs with the same special matrix)*

Let \mathfrak{h}_1 and \mathfrak{h}_2 be the two Hoffman graphs shown in Figure 1. They have both $\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$ as their special matrix, but they are not isomorphic.

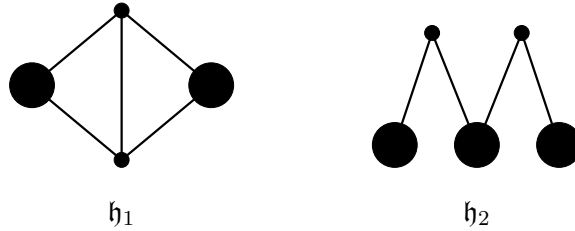


Figure 1

Lemma 2.9. *Let S be the special matrix of a Hoffman graph \mathfrak{h} . Then S satisfies the following properties:*

- (i) S is a symmetric matrix with integral entries;
- (ii) $S_{xx} \leq 0$;
- (iii) $S_{xy} \leq 1$ if $x \neq y$;
- (iv) $S_{xx} \leq S_{xy}$ for all x, y .

The converse of this lemma is not true. For example, there is no Hoffman graph with special matrix $\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$.

Let $\lambda_{\min}(\mathfrak{h})$ denote the smallest eigenvalue of \mathfrak{h} . Woo and Neumaier [12, Corollary 3.3] showed the following result:

Lemma 2.10. *If \mathfrak{g} is an induced Hoffman subgraph of a Hoffman graph \mathfrak{h} , then $\lambda_{\min}(\mathfrak{g}) \geq \lambda_{\min}(\mathfrak{h})$ holds.*

2.4 Direct sums and line Hoffman graphs

Now, we introduce *direct sums* of Hoffman graphs and *line Hoffman graphs*. Note that our definition of direct sums is different from the original definition in [6] (see also [12]). We will show that they are equivalent in Lemma 2.12.

Definition 2.11. *Let \mathfrak{h}^1 and \mathfrak{h}^2 be two Hoffman graphs. We call a Hoffman graph \mathfrak{h} the direct sum of \mathfrak{h}^1 and \mathfrak{h}^2 , denoted by $\mathfrak{h} = \mathfrak{h}^1 \oplus \mathfrak{h}^2$, if \mathfrak{h} satisfies the following condition:*

There exists a partition $\{V_s^1(\mathfrak{h}), V_s^2(\mathfrak{h})\}$ of $V_s(\mathfrak{h})$ such that induced Hoffman subgraphs generated by $V_s^i(\mathfrak{h})$ are \mathfrak{h}^i for $i = 1, 2$ and $S(\mathfrak{h}) = \begin{pmatrix} S(\mathfrak{h}^1) & 0 \\ 0 & S(\mathfrak{h}^2) \end{pmatrix}$ with respect to the partition $\{V_s^1(\mathfrak{h}), V_s^2(\mathfrak{h})\}$ of $V_s(\mathfrak{h})$.

Clearly, by definition, the direct sum is associative, so that the sum $\bigoplus_{i=1}^r \mathfrak{h}^i$ is well-defined. We can check that \mathfrak{h} is a direct sum of two non-empty Hoffman graphs if and only if $S(\mathfrak{h})$ is a block matrix with at least 2 blocks. If $\mathfrak{h} = \mathfrak{h}^1 \oplus \mathfrak{h}^2$ for some non-empty Hoffman subgraphs \mathfrak{h}^1 and \mathfrak{h}^2 , then we call \mathfrak{h} *decomposable*. Otherwise, \mathfrak{h} is called *indecomposable*.

The following lemma gives a combinatorial way to define the direct sum of Hoffman graphs.

Lemma 2.12. *Let \mathfrak{h} be a Hoffman graph and \mathfrak{h}^1 and \mathfrak{h}^2 be two induced Hoffman subgraphs of \mathfrak{h} . The Hoffman graph \mathfrak{h} is the direct sum of \mathfrak{h}^1 and \mathfrak{h}^2 if and only if \mathfrak{h}^1 , \mathfrak{h}^2 , and \mathfrak{h} satisfy the following conditions:*

- (i) $V(\mathfrak{h}) = V(\mathfrak{h}^1) \cup V(\mathfrak{h}^2)$;
- (ii) $\{V_s(\mathfrak{h}^1), V_s(\mathfrak{h}^2)\}$ is a partition of $V_s(\mathfrak{h})$;
- (iii) if $x \in V_s(\mathfrak{h}^i)$, $f \in V_f(\mathfrak{h})$ and $x \sim f$, then $f \in V_f(\mathfrak{h}^i)$;
- (iv) if $x \in V_s(\mathfrak{h}^1)$ and $y \in V_s(\mathfrak{h}^2)$, then x and y have at most one common fat neighbor, and they have one if and only if they are adjacent.

Proof. By direct verification. □

- Example 2.13.** (i) For Hoffman graphs \mathfrak{h}' and \mathfrak{h}'' with $V(\mathfrak{h}') \cap V(\mathfrak{h}'') = \emptyset$, the direct sum $\mathfrak{h}' \oplus \mathfrak{h}''$ is just the disjoint union of \mathfrak{h}' and \mathfrak{h}'' .
- (ii) Let \mathfrak{h}_3 , \mathfrak{h}_4 and \mathfrak{h}_5 be the Hoffman graphs shown in Figure 2. Clearly, \mathfrak{h}_4 and \mathfrak{h}_5 are isomorphic as Hoffman graphs.

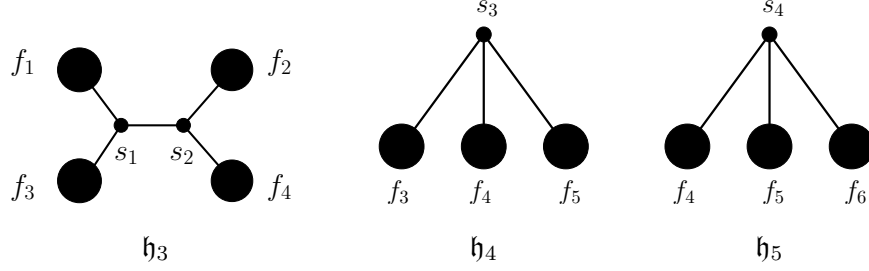


Figure 2

Then $\mathfrak{h}_3 \oplus \mathfrak{h}_4$ and $\mathfrak{h}_3 \oplus \mathfrak{h}_5$ are as in Figure 3. But $\mathfrak{h}_3 \oplus \mathfrak{h}_4$ is not isomorphic to $\mathfrak{h}_3 \oplus \mathfrak{h}_5$.

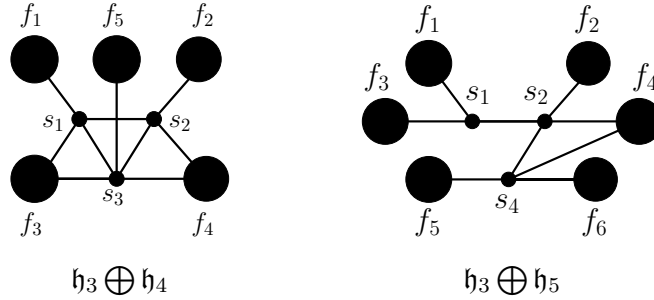


Figure 3

These examples show that there are several ways to construct a direct sum of two Hoffman graphs.

Definition 2.14. Let \mathfrak{H} be a family of Hoffman graphs. A Hoffman graph \mathfrak{h} is called a \mathfrak{H} -line Hoffman graph if \mathfrak{h} is an induced Hoffman subgraph of Hoffman graph $\mathfrak{h}' = \bigoplus_{i=1}^r \mathfrak{h}'_i$ such that they have the same slim graph, where \mathfrak{h}'_i is isomorphic to an induced Hoffman subgraph of some Hoffman graph in \mathfrak{H} for $i = 1, \dots, r$.

Definition 2.15. Let t be a positive integer. The Hoffman graph with one slim vertex adjacent to t fat vertices, denoted by $\mathfrak{h}^{(t)}$, is the unique Hoffman graph with special matrix $(-t)$.

Lemma 2.16. Let t be a positive integer. The t -fat Hoffman graph \mathfrak{h} with special matrix $S(\mathfrak{h}) = J_n - (t+1)I_n$ is unique and this Hoffman graph is a $\{\mathfrak{h}^{(t+1)}\}$ -line Hoffman graph, where J_n is the $n \times n$ all-one matrix.

Proof. Uniqueness is clear because the only possible Hoffman graph for \mathfrak{h} is a Hoffman graph such that its slim graph is K_n , and every slim vertex has its own t fat neighbors. By adding one fat vertex which is adjacent to all slim vertices, we obtain a new Hoffman graph which can be decomposed as the sum of $\mathfrak{h}^{(t+1)}$'s. Hence, \mathfrak{h} is a $\{\mathfrak{h}^{(t+1)}\}$ -line Hoffman graph. \square

Hoffman and Ostrowski showed the following result. For a proof, see [6, Theorem 2.14].

Theorem 2.17. Suppose \mathfrak{h} is a Hoffman graph, and $f_1, \dots, f_r \in V_f(\mathfrak{h})$. Let $\mathfrak{g}^{n_1, \dots, n_r}$ be the Hoffman graph obtained from \mathfrak{h} by replacing each f_i by a slim n_i -clique K^i , and joining all the neighbors of f_i with all the vertices of K^i for all i by edges. Then

$$\lambda_{\min}(\mathfrak{g}^{n_1, \dots, n_r}) \geq \lambda_{\min}(\mathfrak{h}),$$

and

$$\lim_{n_1, \dots, n_r \rightarrow \infty} \lambda_{\min}(\mathfrak{g}^{n_1, \dots, n_r}) = \lambda_{\min}(\mathfrak{h}).$$

3 A structural theorem

In this section, we will obtain a structural theorem for t -fat Hoffman graphs with smallest eigenvalue at least $-t-1$. In Section 3.1, we will consider a set of matrices whose smallest eigenvalue is at most $-1-\sqrt{2}$. In Section 3.2, we will find a finite family $\mathfrak{G}(t)$ of t -fat Hoffman graphs. In Corollary 3.8, we will show that the t -fat Hoffman graphs with smallest eigenvalue at least $-t-1$ are $\mathfrak{G}(t)$ -line Hoffman graphs.

3.1 Forbidden matrices

For two matrices B_1 and B_2 , we say that B_1 is equivalent to B_2 if there exists a permutation matrix P such that $P^T B_1 P = B_2$.

Definition 3.1. Let t and a be two integers where $t > 0$. We define the matrices $m_{1,a}(t)$, $m_{2,a}(t)$, $m_{3,a}(t)$, $m_{4,a}(t)$, $m_5(t)$, $m_6(t)$, $m_7(t)$, $m_8(t)$, $m_9(t)$ as follows:

$$\begin{aligned}
\text{(i)} \quad & m_{1,a}(t) = (-t + a); \\
\text{(ii)} \quad & m_{2,a}(t) = \begin{pmatrix} -t & a \\ a & -t \end{pmatrix}; \\
\text{(iii)} \quad & m_{3,a}(t) = \begin{pmatrix} -t-1 & a \\ a & -t \end{pmatrix}; \\
\text{(iv)} \quad & m_{4,a}(t) = \begin{pmatrix} -t-1 & a \\ a & -t-1 \end{pmatrix}; \\
\text{(v)} \quad & m_5(t) = \begin{pmatrix} -t & -1 & -1 \\ -1 & -t & -1 \\ -1 & -1 & -t \end{pmatrix}, \quad m_6(t) = \begin{pmatrix} -t & 1 & 1 \\ 1 & -t & -1 \\ 1 & -1 & -t \end{pmatrix}, \\
& m_7(t) = \begin{pmatrix} -t & 0 & 1 \\ 0 & -t & -1 \\ 1 & -1 & -t \end{pmatrix}, \quad m_8(t) = \begin{pmatrix} -t & 0 & 1 \\ 0 & -t & 1 \\ 1 & 1 & -t \end{pmatrix}, \\
& m_9(t) = \begin{pmatrix} -t & 0 & -1 \\ 0 & -t & -1 \\ -1 & -1 & -t \end{pmatrix}.
\end{aligned}$$

Definition 3.2. Let t be a positive integer. We define the set $\mathcal{M}(t)$ of matrices as the union of the sets $M_1(t)$, $M_2(t)$, $M_3(t)$, $M_4(t)$ and $M_5(t)$, where

$$\begin{aligned}
M_1(t) &= \{m_{1,a}(t) \mid a = -2, -3, \dots\}, \\
M_2(t) &= \{m_{2,a}(t) \mid a = -2, -3, \dots, -t\}, \\
M_3(t) &= \{m_{3,a}(t) \mid a = 1, -1, -2, \dots, -t\}, \\
M_4(t) &= \{m_{4,a}(t) \mid a = 1, -1, -2, \dots, -t-1\}, \\
M_5(t) &= \{m_5(t), m_6(t), m_7(t), m_8(t), m_9(t)\}.
\end{aligned}$$

Proposition 3.3. Let t be a positive integer and \mathfrak{h} be an indecomposable t -fat Hoffman graph. If special matrix $S(\mathfrak{h})$ contains a principal submatrix

P with $\lambda_{\min}(P) \leq -t - \sqrt{2}$, then $\lambda_{\min}(\mathfrak{h}) \leq -t - \sqrt{2}$. In particular, if $S(\mathfrak{h})$ contains a principal submatrix equivalent to one of the matrices in $\mathcal{M}(t)$, then $\lambda_{\min}(\mathfrak{h}) \leq -t - \sqrt{2}$.

Note that

$$\lambda_{\min}(m_{1,a}(t)) = -t + a, \quad \lambda_{\min}(m_{2,a}(t)) = -t - |a|,$$

$$\lambda_{\min}(m_{3,a}(t)) = -t - \frac{1 + \sqrt{1 + a^2}}{2},$$

$$\lambda_{\min}(m_{4,a}(t)) = -t - 1 - |a|, \quad \lambda_{\min}(m_5(t)) = \lambda_{\min}(m_6(t)) = -t - 2,$$

$$\lambda_{\min}(m_7(t)) = \lambda_{\min}(m_8(t)) = \lambda_{\min}(m_9(t)) = -t - \sqrt{2}.$$

It means that all of the matrices in $\mathcal{M}(t)$ has smallest eigenvalue at most $-t - \sqrt{2}$. Then the proposition follows by Lemma 2.3 (i).

3.2 A family of Hoffman graphs

In this subsection, we will define the family $\mathfrak{G}(t)$ of t -fat Hoffman graphs.

Theorem 3.4. *Let t be a positive integer. Suppose \mathfrak{h} is an indecomposable t -fat Hoffman graph such that its special matrix $S(\mathfrak{h})$ contains none of the principal submatrices equivalent to any element of $\mathcal{M}(t)$. Then $\lambda_{\min}(\mathfrak{h}) \geq -t - 1$ and $S(\mathfrak{h})$ is one of the following:*

- (i) $(-t)$, with smallest eigenvalue $-t$;
- (ii) $(-t - 1)$, with smallest eigenvalue $-t - 1$;
- (iii) $J - (t + 1)I$, with smallest eigenvalue $-t - 1$;
- (iv) $\begin{pmatrix} J - (t + 1)I & -J \\ -J & J - (t + 1)I \end{pmatrix}$, with smallest eigenvalue $-t - 1$.

Moreover, if $S(\mathfrak{h}) = \begin{pmatrix} J_{s_1} - (t + 1)I_{s_1} & -J \\ -J & J_{s_2} - (t + 1)I_{s_2} \end{pmatrix}$, then both s_1 and s_2 are at most t . In particular, $|V_s(\mathfrak{h})| \leq 2t$.

Proof. Suppose that there exists a vertex $x \in V_s(\mathfrak{h})$ such that $S(\mathfrak{h})_{xx} \leq -t-1$. Since \mathfrak{h} is indecomposable and by Lemma 2.9 (iv) and Proposition 3.3, we find that $S(\mathfrak{h})$ does not contain $m_{1,a}(t)$, $m_{3,b}(t)$ or $m_{4,b}(t)$ as a principle submatrix for $a = -2, -3, \dots$ and $b = 1, -1, -2, \dots$. Therefore, we find that $S(\mathfrak{h}) = (-t-1)$ and hence $\mathfrak{h} = \mathfrak{h}^{(t+1)}$ with smallest eigenvalue $-t-1$.

So now, we may assume that $S(\mathfrak{h})_{xx} = -t$ for all $x \in V_s(\mathfrak{h})$. For distinct vertices x and y , it is easy to see that $S(\mathfrak{h})_{xy} \in \{0, 1, -1\}$ since $m_{2,a}(t)$ is not a principal submatrix of $S(\mathfrak{h})$ for $a = -2, -3, \dots$. Considering that \mathfrak{h} is indecomposable and $S(\mathfrak{h})$ does not contain any principal submatrix equivalent to $m_7(t), m_8(t)$ or $m_9(t)$, we obtain that $S(\mathfrak{h})_{xy} \neq 0$. It follows that $S(\mathfrak{h})_{xy} \in \{1, -1\}$.

We define a relation \mathcal{R} on $V_s(\mathfrak{h})$ by $x\mathcal{R}y$ if $(S(\mathfrak{h}))_{xy} = 1$ or $x = y$ holds.

(Claim I) The relation \mathcal{R} on $V_s(\mathfrak{h})$ is an equivalence relation.

(Proof of Claim I) Clearly, \mathcal{R} is reflexive and symmetric. For transitivity, suppose that $x\mathcal{R}y$ and $y\mathcal{R}z$ both hold. If $(S(\mathfrak{h}))_{xz} = -1$, $S(\mathfrak{h})$ contains a principal submatrix which is equivalent to $m_6(t)$. This gives a contradiction. Hence, $(S(\mathfrak{h}))_{xz} = 1$ and $x\mathcal{R}z$ holds.

(Claim II) The number of equivalence classes under \mathcal{R} is at most 2. If $V_s(\mathfrak{h})$ has two equivalence classes, then each equivalence class has size at most t .

(Proof of Claim II) Suppose that there are three equivalence classes. Take three vertices, one vertex from each class. Then, the principal submatrix indexed by these three vertices is the matrix $m_5(t)$, which is a contradiction. Hence, there are at most 2 equivalence classes under \mathcal{R} .

Suppose that $V_s(\mathfrak{h})$ has two equivalence classes C_1 and C_2 . Let $C_1 = \{x_1, x_2, \dots, x_{s_1}\}$. Then for any vertex $y \in C_2$, the vertices x_i and y have at least one common fat neighbor. Note that x_i and x_j have no common fat neighbor if $i \neq j$. It follows that the size of C_1 is at most the number of fat neighbors of y , that is, $s_1 \leq t$. Similarly, we find that $s_2 = |C_2| \leq t$.

If $V_s(\mathfrak{h})$ has only one equivalence class under \mathcal{R} , then $S(\mathfrak{h}) = J - (t+1)I$ and we are in the case (i) or (iii) and $\lambda_{\min}(\mathfrak{h}) \geq -t-1$. If $V_s(\mathfrak{h})$ has two equivalence classes under \mathcal{R} , then $S(\mathfrak{h})$ is of the form

$$S(\mathfrak{h}) = \begin{pmatrix} J - (t+1)I & -J \\ -J & J - (t+1)I \end{pmatrix}$$

in a labeling with respect to the two equivalence classes and $\lambda_{\min}(\mathfrak{h}) = -t-1$. In this case, each class has at most t elements. \square

If an indecomposable t -fat Hoffman graph \mathfrak{h} satisfies the assumption of Theorem 3.4, then \mathfrak{h} has smallest eigenvalue at least $-t - 1$. Conversely, for an indecomposable Hoffman graph \mathfrak{h} with smallest eigenvalue at least $-t - 1$, $S(\mathfrak{h})$ does not contain a principal submatrix which is equivalent to an element of $\mathcal{M}(t)$. Hence, we obtain the corollary below.

Corollary 3.5. *Let \mathfrak{h} be a t -fat Hoffman graph. Then the special matrix $S(\mathfrak{h})$ does not contain any principal submatrix which is equivalent to an element of $\mathcal{M}(t)$ if and only if \mathfrak{h} has smallest eigenvalue at least $-t - 1$.*

Definition 3.6. *Let t be a positive integer. We define $\mathfrak{G}(t)$ to be the family of pairwise non-isomorphic indecomposable t -fat Hoffman graphs with special matrix either $(-t - 1)$ or $\begin{pmatrix} J_{s_1} - (t+1)I_{s_1} & -J \\ -J & J_{s_2} - (t+1)I_{s_2} \end{pmatrix}$, where $1 \leq s_1, s_2 \leq t$.*

Note that $\mathfrak{G}(t)$ is a finite family of Hoffman graphs and the Hoffman graph $\mathfrak{h}^{(t+1)}$ given in Definition 2.15 belongs to $\mathfrak{G}(t)$.

Theorem 3.4 shows the following result, which relates $\mathcal{M}(t)$ to $\mathfrak{G}(t)$.

Theorem 3.7. *Let t be a positive integer and \mathfrak{h} be a t -fat Hoffman graph such that its special matrix $S(\mathfrak{h})$ does not contain any principal submatrix equivalent to an element of $\mathcal{M}(t)$. Then \mathfrak{h} is a t -fat $\mathfrak{G}(t)$ -line Hoffman graph.*

Proof. Let $\mathfrak{h} = \bigoplus_{i=1}^r \mathfrak{h}_i$, where \mathfrak{h}_i is an indecomposable induced Hoffman subgraph of \mathfrak{h} for all i . Without loss of generality, it suffices to show that \mathfrak{h}_1 is a t -fat $\mathfrak{G}(t)$ -line Hoffman graph. Clearly, \mathfrak{h}_1 is t -fat and $S(\mathfrak{h}_1)$ does not contain a principal submatrix equivalent to an element of $\mathcal{M}(t)$. By Theorem 3.4, we have to consider the following 3 cases for $S(\mathfrak{h}_1)$:

- (a) If $S(\mathfrak{h}_1) = (-t)$ or $(-t - 1)$, then $\mathfrak{h}_1 \in \{\mathfrak{h}^{(t)}, \mathfrak{h}^{(t+1)}\}$;
- (b) If $S(\mathfrak{h}_1) = J - (t+1)I$, then \mathfrak{h}_1 is a $\{\mathfrak{h}^{(t+1)}\}$ -line Hoffman graph by Lemma 2.16;
- (c) If $S(\mathfrak{h}_1) = \begin{pmatrix} J - (t+1)I & -J \\ -J & J - (t+1)I \end{pmatrix}$, then $\mathfrak{h}_1 \in \mathfrak{G}(t)$.

Since that $\mathfrak{h}^{(t+1)} \in \mathfrak{G}(t)$ and $\mathfrak{h}^{(t)}$ is an induced Hoffman subgraph of $\mathfrak{h}^{(t+1)}$, we are done. \square

By Corollary 3.5 and Theorem 3.7, we have

Corollary 3.8. *Let t be a positive integer and \mathfrak{h} be a t -fat Hoffman graph with smallest eigenvalue at least $-t - 1$. Then \mathfrak{h} is a t -fat $\mathfrak{S}(t)$ -line Hoffman graph.*

Remark 3.9. *The Corollary 3.8 also holds for t -fat Hoffman graphs with smallest eigenvalue greater than $-t - \sqrt{2}$.*

4 Associated Hoffman graphs

In this section, we first summarize some facts about associated Hoffman graphs and quasi-cliques as defined in [7]. We also will show that under some conditions, the associated Hoffman graph is a line Hoffman graph of a finite family of Hoffman graphs.

Let m be a positive integer and let G be a graph that does not contain \tilde{K}_{2m} as an induced subgraph, where \tilde{K}_{2m} is the graph on $2m + 1$ vertices consisting of a complete graph K_{2m} and a vertex ∞ which is adjacent to exactly m vertices of the K_{2m} . Let $n \geq (m + 1)^2$ be a positive integer. Let $\mathcal{C}(n) := \{C \mid C \text{ is a maximal clique of } G \text{ with at least } n \text{ vertices}\}$. Define the relation \equiv_n^m on $\mathcal{C}(n)$ by $C_1 \equiv_n^m C_2$ if each vertex $x \in C_1$ has at most $m - 1$ non-neighbours in C_2 and each vertex $y \in C_2$ has at most $m - 1$ non-neighbours in C_1 . Then \equiv_n^m is an equivalence relation.

Let $[C]_n^m$ denote the equivalence class of $\mathcal{C}(n)$ of G under the equivalence relation \equiv_n^m containing the maximal clique C of $\mathcal{C}(n)$. We define the quasi-clique with respect to the pair (m, n) , $Q([C]_n^m)$, as the induced subgraph of G on the set $\{x \in V(G) \mid x \text{ has at most } m - 1 \text{ non-neighbours in } C\}$.

Let $[C_1]_n^m, \dots, [C_r]_n^m$ be equivalence classes of maximal cliques under \equiv_n^m . The *associated Hoffman graph* $\mathfrak{g} = \mathfrak{g}(G, m, n)$ is the Hoffman graph satisfying the following conditions:

- (i) $V_s(\mathfrak{g}) = V(G)$, $V_f(\mathfrak{g}) = \{F_1, \dots, F_r\}$;
- (ii) The slim graph of \mathfrak{g} equals G ;
- (iii) The fat vertex F_i is adjacent to exactly all the vertices of $Q([C_i]_n^m)$ for $i = 1, 2, \dots, r$.

Let \mathfrak{h} be a Hoffman graph with $V_f(\mathfrak{h}) = \{F_1, \dots, F_r\}$ for some positive integer r . The graph $G(\mathfrak{h}, n)$ is the slim graph of the Hoffman graph $\mathfrak{g}^{n_1, \dots, n_r}$ (as defined in Theorem 2.17) with $n_i = n$ for $i = 1, \dots, r$.

The following result is shown in [7, Proposition 4.1].

Proposition 4.1. *Let G be a graph and let $m \geq 2, \phi \geq 1, \sigma \geq 1, p \geq 1$ be integers. There exists a positive integer $n = n(m, \phi, \sigma, p) \geq (m+1)^2$ such that for any integer $q \geq n$, and any Hoffman graph \mathfrak{h} with at most ϕ fat vertices and at most σ slim vertices, the graph $G(\mathfrak{h}, p)$ is an induced subgraph of G , provided that the graph G satisfies the following conditions:*

- (i) *The graph G does not contain \tilde{K}_{2m} as an induced subgraph;*
- (ii) *Its associated Hoffman graph $\mathfrak{g} = \mathfrak{g}(G, m, q)$ contains \mathfrak{h} as an induced Hoffman subgraph.*

The theorem below is a modification of [7, Theorem 6.2] and its proof follows the proof of [7, Theorem 6.2]. We give it for the convenience of readers. It will be used in the next section.

Theorem 4.2. *Let t be a positive integer and $m(t) = \min\{m \mid \lambda_{\min}(\tilde{K}_{2m}) < -t - 1\}$. Then there exists a positive integer $p(t)$ such that if G is a graph with $\lambda_{\min}(G) \geq -t - 1$, then for any integer $r \geq p(t)$, the quasi-cliques Q_1, Q_2, \dots, Q_c of G with respect to the pair $(m(t), r)$ satisfy the following conditions:*

- (i) *The complement of $V(Q_i)$ has degree at most t^2 for $i = 1, 2, \dots, c$;*
- (ii) *The intersection $V(Q_i) \cap V(Q_j)$ contains at most t vertices for $1 \leq i < j \leq c$.*

Proof. Let H be a graph with $t^2 + 2$ vertices such that at least one vertex has degree $t^2 + 1$. Then, by the Perron-Frobenius theorem ([3], Theorem 8.8.1 (b)), the largest eigenvalue of H is at least $\sqrt{t^2 + 1} > t$. Now, consider the fat Hoffman graph $\mathfrak{b} = \mathfrak{b}(H)$ with one fat vertex such that the slim graph of $\mathfrak{b}(H)$ is equal to the complement of H . Then $S(\mathfrak{b}) = -A(H) - I$. It follows that $\lambda_{\min}(\mathfrak{b}) \leq -\sqrt{t^2 + 1} - 1 < -t - 1$. Hence, by Theorem 2.17 and Proposition 4.1, there exists a positive integer $p_H \geq (m(t) + 1)^2$ such that for any integer $r_1 \geq p_H$, the associated Hoffman graph $\mathfrak{g}(G, m(t), r_1)$ does not contain $\mathfrak{b}(H)$ as an induced Hoffman subgraph. Let p_1 be the maximum of all p_H , where the maximum is taken over all graphs H having exactly $t^2 + 2$ vertices and containing a vertex with degree $t^2 + 1$.

Let \mathfrak{h} be any fat Hoffman graph with exactly two fat vertices F_1, F_2 and $t + 1$ slim vertices x_1, x_2, \dots, x_{t+1} , such that each vertex x_i is adjacent to F_1 and F_2 , for $i = 1, 2, \dots, t + 1$. Then the diagonal elements of $S(\mathfrak{h})$ are equal

to -2 and the rest of the entries of $S(\mathfrak{h})$ are in $\{-2, -1\}$. This shows that $\lambda_{\min}(\mathfrak{h}) \leq -t - 2 < -t - 1$. Hence, by Theorem 2.17 and Proposition 4.1, there exists a positive integer $p(\mathfrak{h}) \geq (m(t) + 1)^2$ such that for any integer $r_2 \geq p(\mathfrak{h})$, the associated Hoffman graph $\mathfrak{g}(G, m(t), r_2)$ does not contain \mathfrak{h} as an induced Hoffman subgraph. Let p_2 be the maximum of all such $p(\mathfrak{h})$'s.

Now take $p(t) = \max\{p_1, p_2\}$. Then the result follows. \square

Now, by using Proposition 4.1, we can show the theorem below.

Theorem 4.3. *Let t be a positive integer and $m(t) = \min\{m \mid \lambda_{\min}(\tilde{K}_{2m}) < -t - 1\}$. Then there exists a positive integer $p'(t)$ such that if G is a graph with $\lambda_{\min}(G) \geq -t - 1$, then for any integer $r \geq p'(t)$, the associated Hoffman graph $\mathfrak{g}(G, m(t), r)$ does not contain an induced Hoffman subgraph whose special matrix is contained in $\mathcal{M}(t)$.*

Moreover, if $\mathfrak{g}(G, m(t), r)$ is t -fat for some $r \geq p'(t)$, then G is the slim graph of a t -fat $\mathfrak{G}(t)$ -line Hoffman graph.

Proof. Let G be a graph with $\lambda_{\min}(G) \geq -t - 1$. First, it is clear that G does not contain $\tilde{K}_{2m(t)}$ as an induced subgraph. We will show that there exists a positive integer $p'(t)$ such that for $r \geq p'(t)$, the associated Hoffman graph $\mathfrak{g}(G, m(t), r)$ does not contain an induced Hoffman subgraph whose special matrix is an element of $\mathcal{M}(t)$. Note that every Hoffman graph whose special matrix is in $M_1(t)$ has only one slim vertex and contains $\mathfrak{h}^{(t+2)}$ as an induced Hoffman subgraph. Let $\mathfrak{h}_1 = \mathfrak{h}^{(t+2)}$ and $\mathfrak{h}_2, \dots, \mathfrak{h}_{t'}$ be all of Hoffman graphs with at least 2 slim vertices whose special matrices are an element of $\mathcal{M}(t)$. Then each \mathfrak{h}_i has at most 3 slim vertices and at most $3t + 1$ fat vertices. Since $\lambda_{\min}(\mathfrak{h}_i) < -t - 1$, there exists a positive integer p_i such that $\lambda_{\min}(G(\mathfrak{h}_i, p_i)) < -t - 1$ for all $1 \leq i \leq t'$ by Theorem 2.17. Let $p'_i = n(m(t), 3t + 1, 3, p_i)$ and $p'(t) = \max_{1 \leq i \leq t'} p'_i$. Then for $r \geq p'(t)$, $\mathfrak{g}(G, m(t), r)$ does not contain an induced Hoffman subgraph whose special matrix is an element of $\mathcal{M}(t)$ by Proposition 4.1. If $\mathfrak{g}(G, m(t), r)$ is t -fat for some $r \geq p'(t)$, $\mathfrak{g}(G, m(t), r)$ is a t -fat $\mathfrak{G}(t)$ -line Hoffman graph by Theorem 3.7. Hence, G is the slim graph of a t -fat $\mathfrak{G}(t)$ -line Hoffman graph. \square

5 Main theorems

Recall that the Ramsey number $R(a, b)$ is defined by the well-known Ramsey's Theorem.

Theorem 5.1. ([8]) *Let a, b be two positive integers. Then there exists a minimum positive integer $R(a, b)$ such that for any graph G on $n \geq R(a, b)$ vertices, the graph G contains a dependent set of size a or an independent set of size b .*

By using the concept of associated Hoffman graph and Ramsey theory, we will prove our main theorems as follows.

Theorem 5.2. *Let $t \geq 2$ be an integer and $s \in \{t-1, t\}$. Then, there exists a positive integer $K(t)$, such that if a graph G satisfies the following conditions:*

- (i) $d(x) > K(t)$ holds for all $x \in V(G)$;
- (ii) For all $x \in V(G)$, any $(t^2 + 1)$ -plex containing x has the order at most $\frac{d(x) - K(t)}{s}$;
- (iii) $\lambda_{\min}(G) \geq -t - 1$,

then the following hold:

- (a) If $s = t - 1$, then G is the slim graph of a t -fat $\mathfrak{S}(t)$ -line Hoffman graph;
- (b) If $s = t$, then G is the slim graph of a $(t + 1)$ -fat $\mathfrak{h}^{(t+1)}$ -line Hoffman graph.

Proof. Let $m(t) = \min\{m \mid \lambda_{\min}(\tilde{K}_{2m}) < -t - 1\}$. Then G does not contain $\tilde{K}_{2m(t)}$ as an induced subgraph. Choose $n(t) = \max\{p(t), p'(t)\}$ where $p(t)$ and $p'(t)$ are such that Theorem 4.2, Theorem 4.3 both hold. Let $K(t) = R(n(t), (t+1)^2 + 1)$, where $R(n(t), (t+1)^2 + 1)$ denotes the Ramsey number.

Pick a vertex x and let $Q_1, \dots, Q_{c(x)}$ be the quasi-cliques with respect to the pair $(m(t), n(t))$ containing x . Suppose that $c(x) \leq s$, then from Theorem 4.2 (i), we know that each Q_i is a $(t^2 + 1)$ -plex, so $|V(Q_i)| \leq \frac{d(x) - K(t)}{s}$ for $i = 1, \dots, c(x)$. We obtain

$$\left| \bigcup_{i=1}^{c(x)} V(Q_i) \right| \leq \sum_{i=1}^{c(x)} |V(Q_i)| \leq \frac{(d(x) - K(t))c(x)}{s} \leq d(x) - K(t).$$

Now, $V(G_x) - \bigcup_i V(Q_i)$ has at least $K(t) = R(n(t), (t+1)^2 + 1)$ vertices and G does not contain $((t+1)^2 + 1)$ -claw as an induced subgraph. Hence we can find a new clique with at least $n(t)$ vertices in $V(G_x) - \bigcup_i V(Q_i)$, and a new

quasi-clique $Q_{c(x)+1}$ containing x . This makes a contradiction, so we obtain $c(x) \geq s + 1$. This implies that each vertex x in G is contained in at least $s + 1$ quasi-cliques with respect to the pair $(m(t), n(t))$ and the associated Hoffman graph $\mathfrak{g} = \mathfrak{g}(G, m(t), n(t))$ is $(s + 1)$ -fat.

If $s = t - 1$, the graph G is the slim graph of a t -fat $\mathfrak{G}(t)$ -line Hoffman graph by Theorem 4.3. This shows (a).

If $s = t$, the graph G is the slim graph of a $(t + 1)$ -fat $\mathfrak{G}(t)$ -line Hoffman graph. However, the only $(t + 1)$ -fat Hoffman graph in $\mathfrak{G}(t)$ is $\mathfrak{h}^{(t+1)}$. Then G is the slim graph of a $(t + 1)$ -fat $\mathfrak{h}^{(t+1)}$ -line Hoffman graph and (b) follows. \square

Proof of Theorem 1.2. Let $K(t)$ be the integer such that Theorem 5.2 holds. Then by using Theorem 5.2, we find that G is the slim graph of some Hoffman graph $\mathfrak{h} = \bigoplus_{i=1}^v \mathfrak{h}_i$ where \mathfrak{h}_i is isomorphic to $\mathfrak{h}^{(t+1)}$ for $i = 1, 2, \dots, v$.

Now define a hypergraph H with vertex set $V_f(\mathfrak{h})$ and (hyper)edge set $\{N_{\mathfrak{h}}^f(x) \mid x \in V_s(\mathfrak{h})\}$. Note that H is a linear $(t + 1)$ -uniform hypergraph. Let $x, y \in V_s(\mathfrak{h})$, then x and y are adjacent in G if and only if $|N_{\mathfrak{h}}^f(x) \cap N_{\mathfrak{h}}^f(y)| = 1$. This shows that G is the intersection graph of H . \square

Proof of Theorem 1.4. Let $K = K(2)$ be the integer such that Theorem 5.2 holds. By using Theorem 5.2, we find that G is the slim graph of a 2-fat $\mathfrak{G}(2)$ -line Hoffman graph. Now we will determine the finite family $\mathfrak{G}(2)$ of Hoffman graphs. For a Hoffman graph \mathfrak{h} in $\mathfrak{G}(2)$,

- 1 If the special matrix of \mathfrak{h} is equal to (-3) , then \mathfrak{h} is the Hoffman graph $\mathfrak{h}^{(3)}$ by Definition 2.15, that is \clubsuit ;
- 2 If the special matrix of \mathfrak{h} is $\begin{pmatrix} J_{s_1} - 3I_{s_1} & -J \\ -J & J_{s_2} - 3I_{s_2} \end{pmatrix}$, where s_1, s_2 are integers such that $1 \leq s_1, s_2 \leq 2$, then \mathfrak{h} will be given as follows:
 - (a) When $s_1 = s_2 = 2$, then \mathfrak{h} is the Hoffman graph \spadesuit ;
 - (b) When $s_1 = 2, s_2 = 1$ or $s_1 = 1, s_2 = 2$, then \mathfrak{h} is the Hoffman graph \heartsuit ;
 - (c) When $s_1 = s_2 = 1$, then \mathfrak{h} is the Hoffman graph \clubsuit or \spadesuit .

So $\mathfrak{G}(2) = \{\text{A}, \text{B}, \text{C}, \text{D}, \text{E}\}$ and G is the slim graph of a 2-fat $\{\text{A}, \text{B}, \text{C}, \text{D}, \text{E}\}$ -line Hoffman graph.

Since B and C are induced Hoffman subgraphs of B , it is easy to check that G is also the slim graph of a 2-fat $\{\text{A}, \text{B}, \text{E}\}$ -line Hoffman graph. \square

Theorem 5.3. *Let $t \geq 2$ be an integer and $s \in \{t-1, t\}$. Then, there exists a positive integer $\kappa(t)$, such that if a graph G satisfies the following conditions:*

- (i) $d(x) > \kappa(t)$ holds for all $x \in V(G)$;
- (ii) For all $x \in V(G)$, $\bar{d}(G_x) \leq \frac{d(x) - \kappa(t)}{s}$ holds;
- (iii) $\lambda_{\min}(G) \geq -t-1$,

then the following hold:

- (a) If $s = t-1$, then G is the slim graph of a t -fat $\mathfrak{G}(t)$ -line Hoffman graph;
- (b) If $s = t$, then G is the slim graph of a $(t+1)$ -fat $\mathfrak{h}^{(t+1)}$ -line Hoffman graph.

Proof. Let $m(t) = \min\{m \mid \lambda_{\min}(\tilde{K}_{2m}) < -t-1\}$, then G does not contain $\tilde{K}_{2m(t)}$ as an induced subgraph. Choose $n(t) = \max\{p(t), p'(t), 2t^2 - t + 2\}$ where $p(t)$ and $p'(t)$ are such that Theorem 4.2, Theorem 4.3 both hold. Let $\kappa(t) = 2R(n(t), (t+1)^2 + 1) + (t^2 + 1)t$, where $R(n(t), (t+1)^2 + 1)$ denotes the Ramsey number. It suffices to show that the associated Hoffman graph $\mathfrak{g}(G, m(t), n(t))$ is $(s+1)$ -fat.

Pick a vertex x and let $Q_1, \dots, Q_{c(x)}$ be the quasi-cliques with respect to the pair $(m(t), n(t))$ containing x . Suppose that $c(x) \leq s$. Let Q'_i be a subgraph of G induced by $(V(G_x) \cap V(Q_i)) - \bigcup_{j < i} V(Q_j)$ and let $\alpha_i = |V(Q'_i)|$ for $i = 1, \dots, c(x)$. By Theorem 4.2 (ii), we know that $|V(Q'_i)| \geq |V(Q_i)| - (i-1)t \geq n(t) - (c(x)-1)t \geq n(t) - (s-1)t > t^2 + 1$. Note that $\sum_{i=1}^{c(x)} \alpha_i = d(x) - d'$ with $d' < R(n(t), (t+1)^2 + 1)$.

Then by using Theorem 4.2 (i)

$$\begin{aligned}
2|E(G_x)| &\geq 2 \sum_{i=1}^{c(x)} |E(Q'_i)| \\
&\geq \sum_{i=1}^{c(x)} \alpha_i (\alpha_i - 1 - t^2) \\
&= \sum_{i=1}^{c(x)} \alpha_i^2 - (t^2 + 1) \sum_{i=1}^{c(x)} \alpha_i \\
&\geq \frac{(\sum_{i=1}^{c(x)} \alpha_i)^2}{c(x)} - (t^2 + 1)(d(x) - d') \\
&\geq \frac{(d(x) - d')^2}{s} - (t^2 + 1)(d(x) - d').
\end{aligned}$$

By (ii), $2|E(G_x)| = \bar{d}(G_x)d(x) \leq \frac{d(x) - \kappa(t)}{s}d(x)$. Now we have

$$\begin{aligned}
0 &\geq \frac{(d(x) - d')^2}{s} - (t^2 + 1)(d(x) - d') - \frac{(d(x) - \kappa(t))d(x)}{s} \\
&= \frac{\kappa(t) - 2d' - (t^2 + 1)s}{s}d(x) + \frac{d'^2}{s} + (t^2 + 1)d'.
\end{aligned}$$

Since $\kappa(t) = 2R(n(t), (t+1)^2 + 1) + (t^2 + 1)t > 2d' + (t^2 + 1)s$, it gives a contradiction.

Hence, we can conclude that the associated Hoffman graph $\mathfrak{g}(G, m(t), n(t))$ is $(s+1)$ -fat. This finishes the proof. \square

Proof of Theorem 1.3. Let $\kappa(t)$ be the integer such that Theorem 5.3 holds. Then by Theorem 5.3, we know that G is the slim graph of a $(t+1)$ -fat $\mathfrak{h}^{(t+1)}$ -line Hoffman graph. As we did in the proof of Theorem 1.2, it is easy to see that G is the intersection graph of some linear $(t+1)$ -uniform hypergraph. \square

Proof of Theorem 1.5. Let $\kappa = \kappa(2)$ be the integer such that Theorem 5.3 holds. Then by Theorem 5.3, we know that G is the slim graph of a 2-fat $\mathfrak{G}(2)$ -line Hoffman graph. Similarly, by the proof of Theorem 1.4, G is the slim graph of a 2-fat $\{\bullet\bullet, \bullet\bullet\bullet, \bullet\bullet\bullet\bullet\}$ -line Hoffman graph. \square

6 Applications

6.1 The graphs cospectral with $H(3, q)$

Let $D, q \geq 2$ be integers. Let X be a finite set of size q . The Hamming graph $H(D, q)$ is the graph with the vertex set $X^D := \prod_{i=1}^D X$ (the Cartesian product of D -copy of X) where two vertices are adjacent whenever they differ in precisely one coordinate. The graph $H(D, q)$ has distinct eigenvalues $\lambda_i = q(D - i) - D$ with multiplicities $m_i = \binom{D}{i}(q - 1)^i$ for $i = 0, 1, \dots, D$. In particular, the Hamming graph $H(3, q)$ has spectrum $\{(3q - 3)^1, (2q - 3)^{3(q-1)}, (q - 3)^{3(q-1)^2}, (-3)^{(q-1)^3}\}$. The following result was first shown by S. Bang et al. [1].

Theorem 6.1. *There exists a positive integer q' such that for all $q \geq q'$, any graph that is cospectral with the Hamming graph $H(3, q)$ is the intersection graph of some linear 3-uniform hypergraph.*

Proof. Let G be a graph cospectral with $H(3, q)$ and let A be the adjacency matrix of G . By (1), we know that for any vertex x in G ,

$$\bar{d}(G_x) = q - 2.$$

Let $q' = \kappa(2) - 1$, where $\kappa(2)$ is an integer such that Theorem 1.3 holds. Then by Theorem 1.3, the result follows. \square

6.2 The graphs cospectral with $J(v, 3)$

Let $v, p \geq 2$ be integers. Let X be a finite set of size v . The Johnson graph $J(v, p)$ is the graph with vertex set $\binom{X}{p}$, the set of all p -subsets of X , where two p -subsets are adjacent if they intersect precisely $p - 1$ elements. The graph $J(v, p)$ has distinct eigenvalues $\lambda_i = (p - i)(v - p - i) - i$ with multiplicities $m_i = \binom{v}{i} - \binom{v}{i-1}$ for $i = 0, 1, \dots, \min\{v - p, p\}$. (Since that the Johnson graph $J(v, p)$ is isomorphic to the Johnson graph $J(v, v - p)$, we always assume that $v \geq 2p$.) In particular, the Johnson graph $J(v, 3)$ has spectrum $\{(3v - 9)^1, (2v - 9)^{v-1}, (v - 7)^{\frac{v(v-3)}{2}}, (-3)^{\frac{v(v-1)(v-5)}{6}}\}$. Now we show the following result.

Theorem 6.2. *There exists a positive integer v' such that for all $v \geq v'$, any graph that is cospectral with the Johnson graph $J(v, 3)$ is the intersection graph of some linear 3-uniform hypergraph.*

Proof. Let G be a graph cospectral with $J(v, 3)$ and let A be the adjacency matrix of G . By (1), we know that for any vertex x in G ,

$$\bar{d}(G_x) = v - 2.$$

Let $v' = \kappa(2) + 5$, where $\kappa(2)$ is an integer such that Theorem 1.3 holds. Then by Theorem 1.3, the result follows. \square

6.3 The graphs cospectral with the 2-clique extension of $(t_1 \times t_2)$ -grid

The graph $(t_1 \times t_2)$ -grid is the line graph of the complete bipartite graph, K_{t_1, t_2} . In other words, it is the graph $K_{t_1} \square K_{t_2}$, where \square represents *Cartesian product*. The spectrum of the $(t_1 \times t_2)$ -grid is $\{(t_1 + t_2 - 2)^1, (t_1 - 2)^{t_2 - 1}, (t_2 - 2)^{t_1 - 1}, (-2)^{(t_1 - 1)(t_2 - 1)}\}$ (See, for example, Section 9.7 of [3]).

A graph \tilde{G} is called the 2-clique extension of graph G if \tilde{G} has matrix $\begin{pmatrix} A - I & A \\ A & A - I \end{pmatrix}$ as its adjacency matrix, where A is the adjacency matrix of G . So, if G has spectrum

$$\{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_t^{m_t}\},$$

then the spectrum of \tilde{G} is

$$\{(2\lambda_0 + 1)^{m_0}, (2\lambda_1 + 1)^{m_1}, \dots, (2\lambda_t + 1)^{m_t}, (-1)^{(m_0 + m_1 + \dots + m_t)}\}.$$

Now we show the following result.

Theorem 6.3. *There exists a positive integer t such that for all $t_1 \geq t_2 \geq t$, any graph that is cospectral with the 2-clique extension of $(t_1 \times t_2)$ -grid is the slim graph of a 2-fat $\{\clubsuit, \heartsuit, \spadesuit, \diamondsuit\}$ -line Hoffman graph.*

Proof. Let G be a graph cospectral with the 2-clique extension of $(t_1 \times t_2)$ -grid with $t_1 \geq t_2$, then the spectrum of G is

$$\{(2t_1 + 2t_2 - 3)^1, (2t_1 - 3)^{t_2 - 1}, (2t_2 - 3)^{t_1 - 1}, (-1)^{t_1 t_2}, (-3)^{(t_1 - 1)(t_2 - 1)}\}.$$

For any vertex x in G , let P be a 5-plex containing x . Then by Lemma 2.4, we know that

$$|V(P)| \leq \frac{2t_1 t_2 (5 + 2t_1 - 3)}{2t_1 t_2 - (2t_1 + 2t_2 - 3) + (2t_1 - 3)} = \frac{2t_1(t_1 + 1)}{t_1 - 1}.$$

Since that $d(x) - \frac{2t_1(t_1+1)}{t_1-1} = \frac{2t_1t_2-7t_1-2t_2+3}{t_1-1} \geq \frac{2t_1t_2-9t_1+3}{t_1} \geq 2t_2 - 9$.

So let $t = \lceil \frac{K+9}{2} \rceil$, where K is an integer such that Theorem 1.4 holds. Then, by using Theorem 1.4, the result follows. □

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